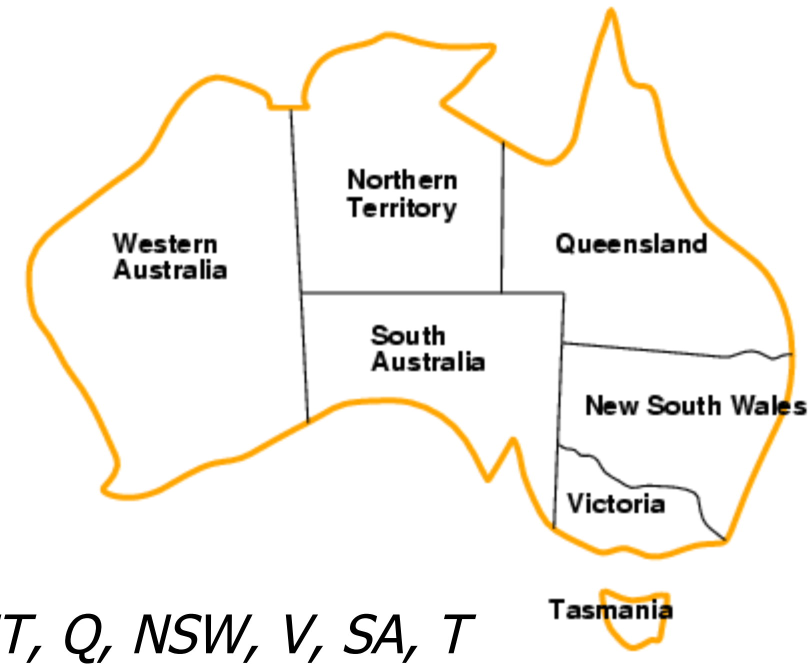
Constraint Satisfaction Problems

Based on Norvig&Russell, Ch. 5

Constraint satisfaction problems (CSPs)

- Standard search problem: state is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- Constraint Satisfaction Problems:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Supports useful general-purpose algorithms with more power than standard search algorithms

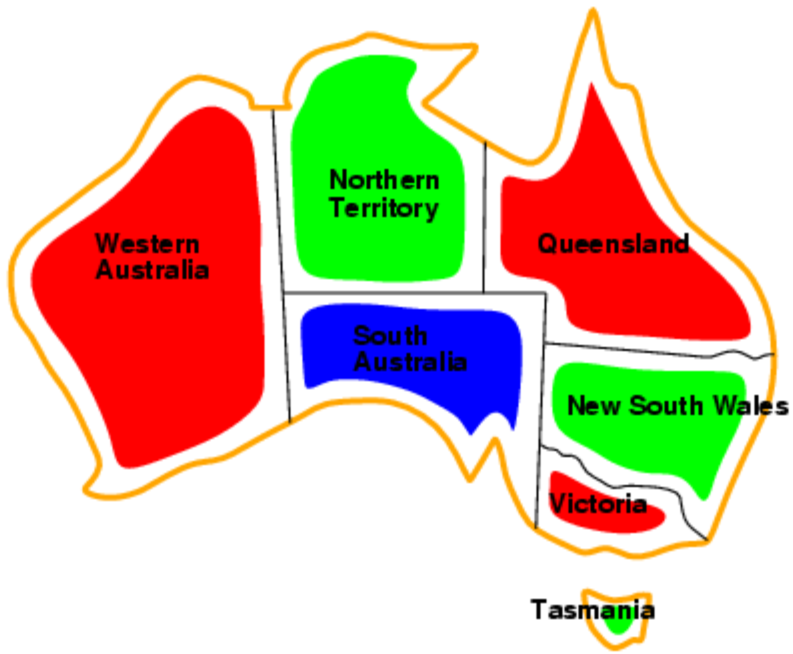
Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors

- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red),(green,blue),(blue,red),(blue,green)}

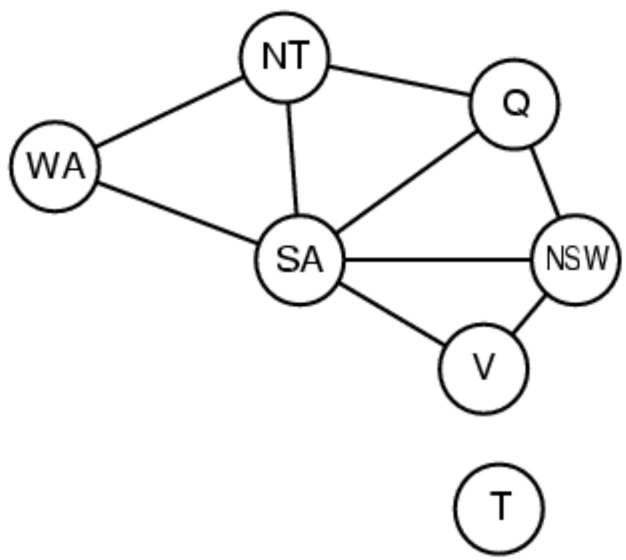
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red, NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

Discrete variables

- finite domains:
  - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
  - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
- infinite domains:
  - integers, strings, etc.
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$

Continuous variables

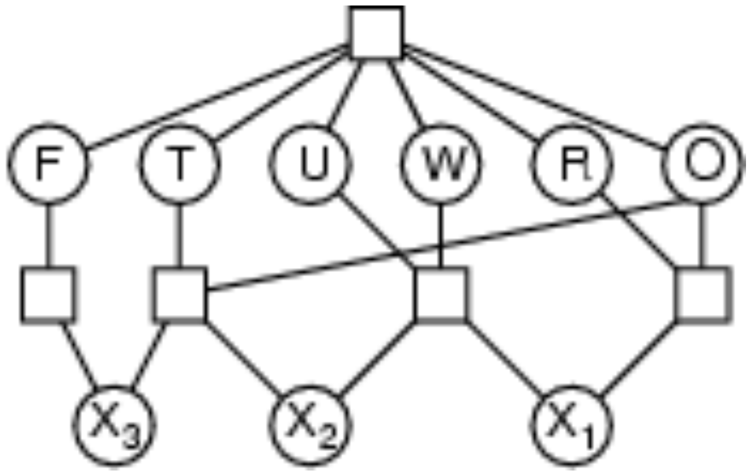
- e.g., start/end times for Hubble Space Telescope observations
- linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

Example: Cryptarithmetic

```
    T W O
  + T W O
  -------
  F O U R
```



Variables: *F T U W R O* X_1 X_2 X_3

Domains: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Constraints: *Alldiff (F, T, U, W, R, O)*

- $O + O = R + 10 \cdot X_1$
- $X_1 + W + W = U + 10 \cdot X_2$
- $X_2 + T + T = O + 10 \cdot X_3$
- $X_3 = F$, $T \neq 0$, $F \neq 0$

Real-world CSPs

Assignment problems

- e.g., who teaches what class

Timetabling problems

- e.g., which class is offered when and where?

Transportation scheduling

Factory scheduling

Many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  → fail if no legal assignments

- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant, so we can also use complete-state formulation
4. $b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves

Backtracking search

- Variable assignments are commutative}

So [WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
  → b = d and there are d^n leaves

- Depth-first search for CSPs with single-variable assignments is called backtracking search

- Backtracking search is the basic uninformed algorithm for CSPs

- Can solve *n*-queens for $n \approx 25$

Backtracking search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Backtracking example

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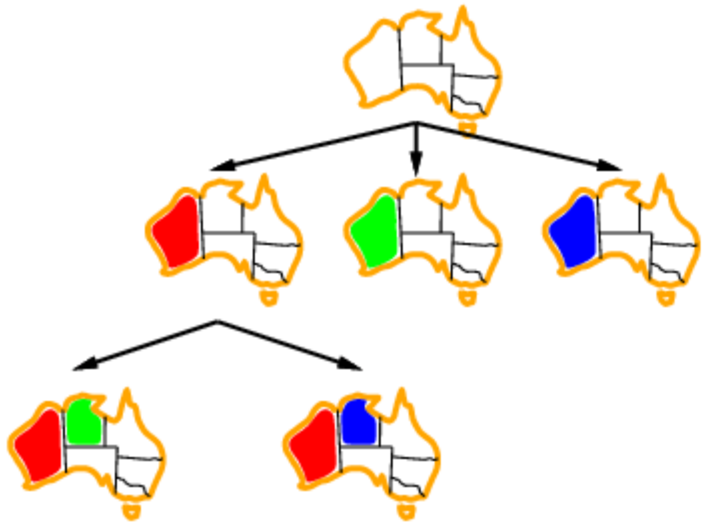
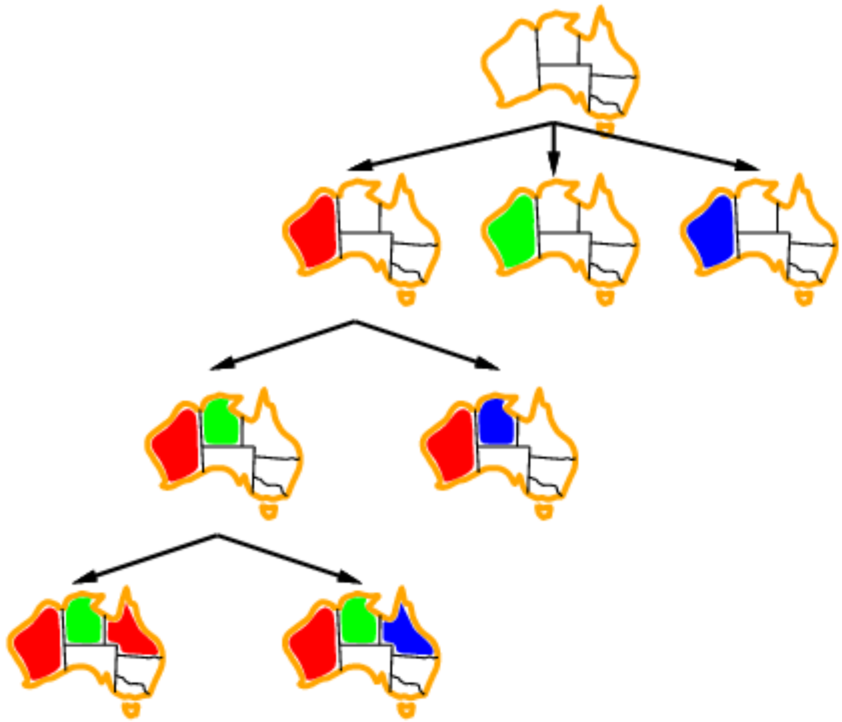


Backtracking example



Improving backtracking efficiency

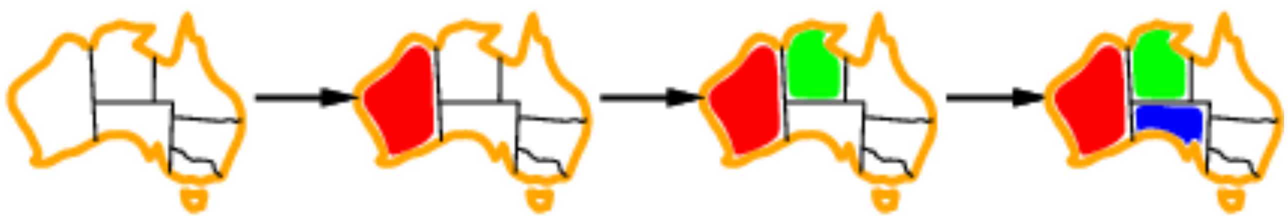
General-purpose methods can give huge gains in speed:

- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?

Most constrained variable

Most constrained variable:

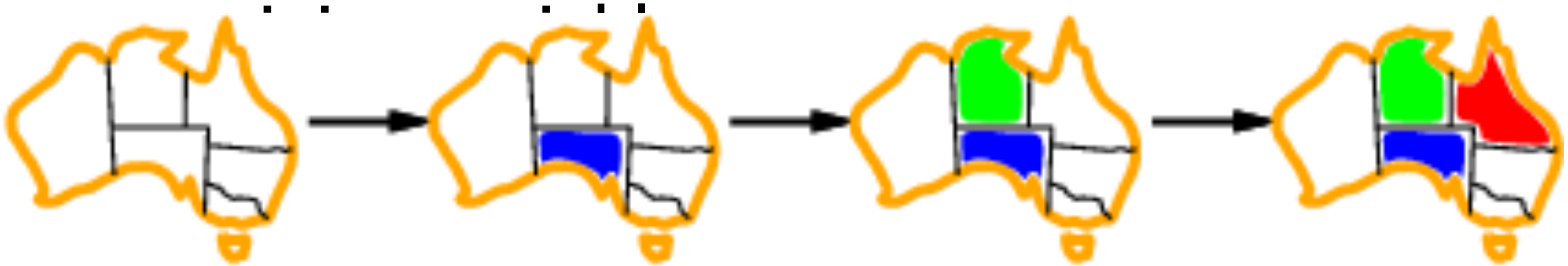
choose the variable with the fewest legal values



a.k.a. minimum remaining values (MRV) heuristic

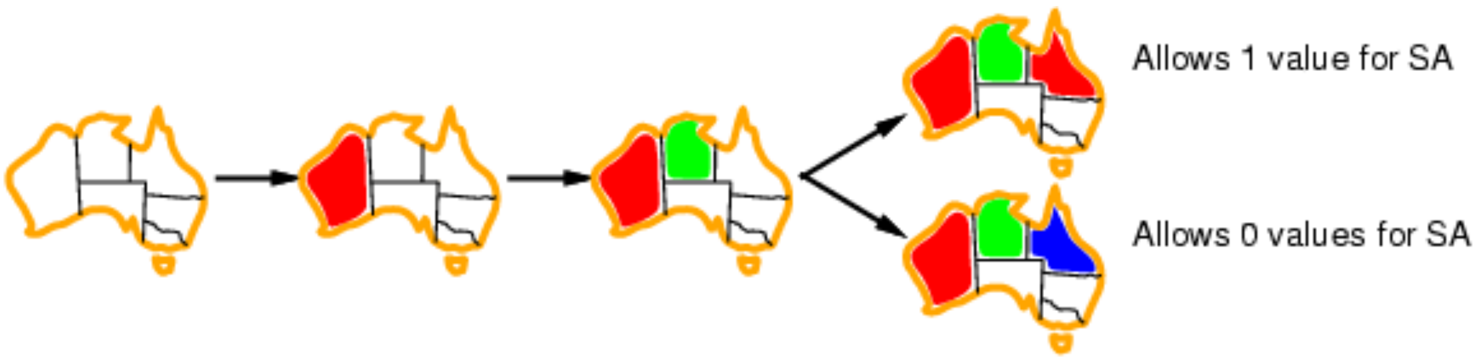
Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on



Least constraining value

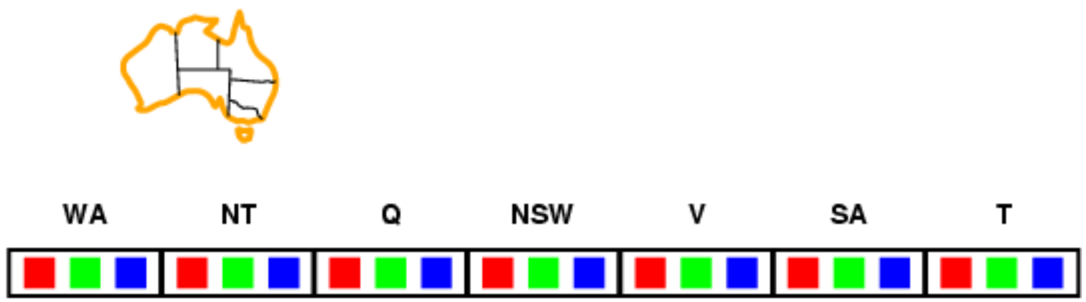
- Given a variable, choose the **least constraining value**:
  the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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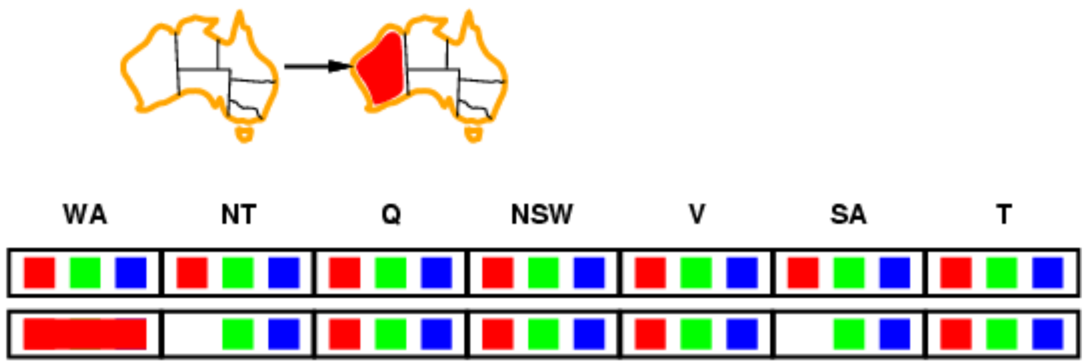

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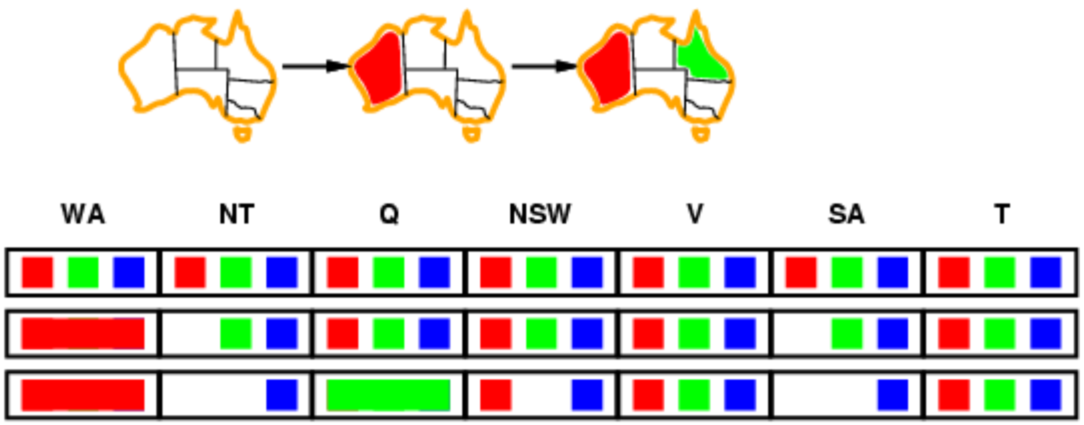

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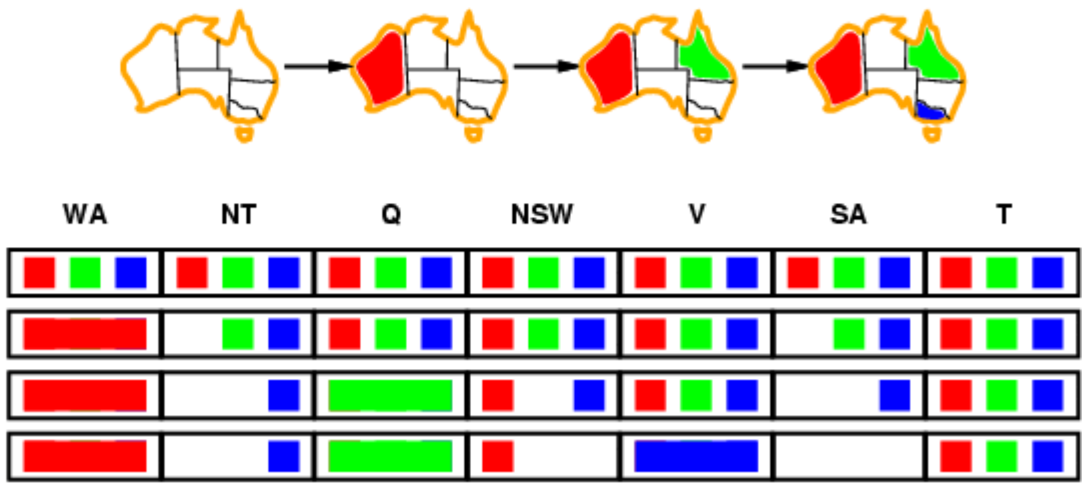
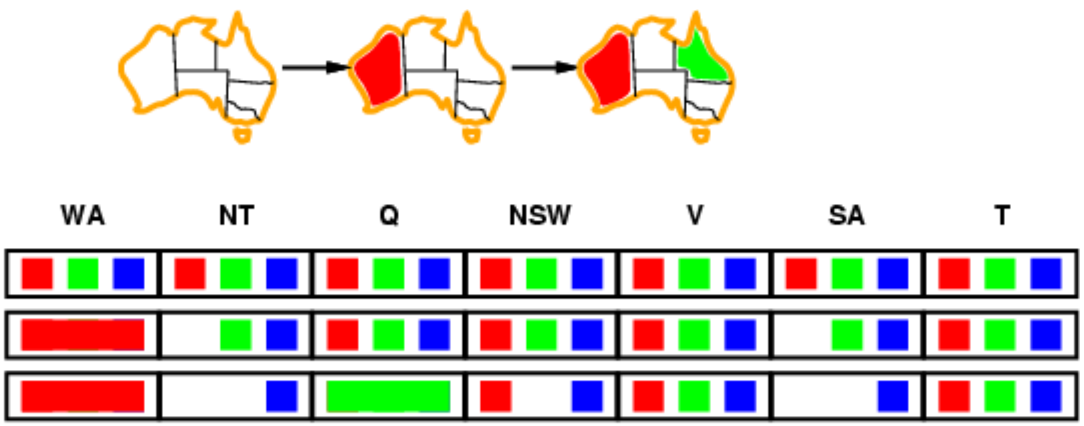


Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



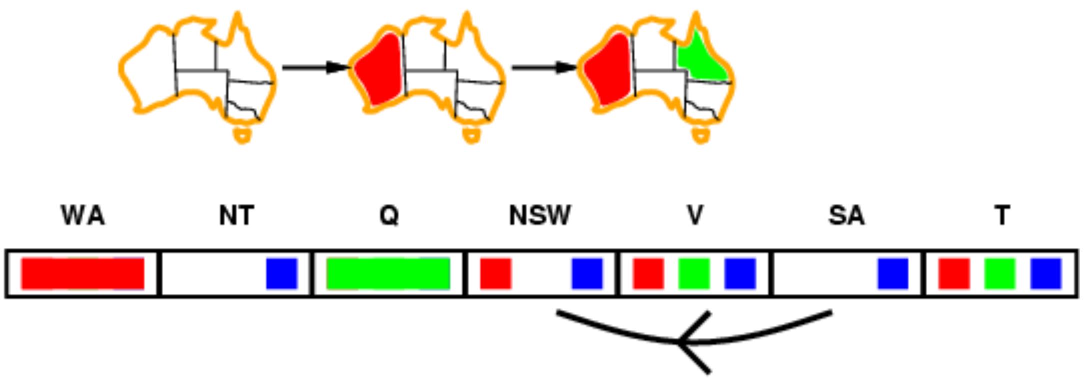
- NT and SA cannot both be blue!

- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



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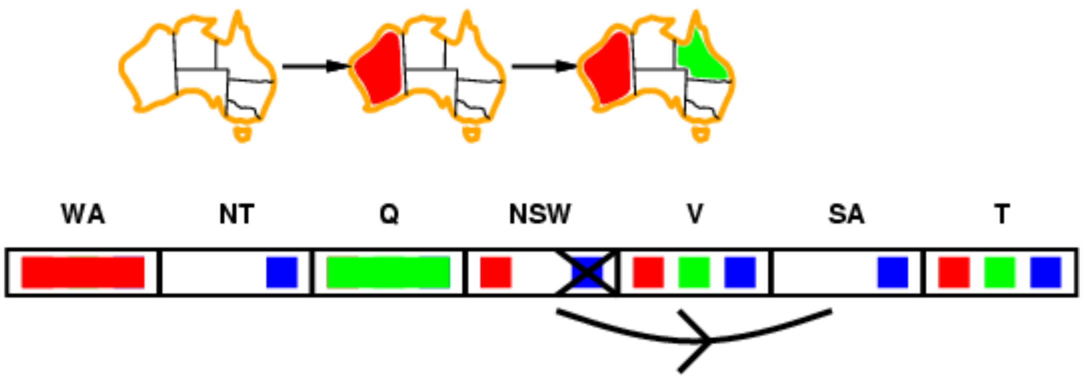
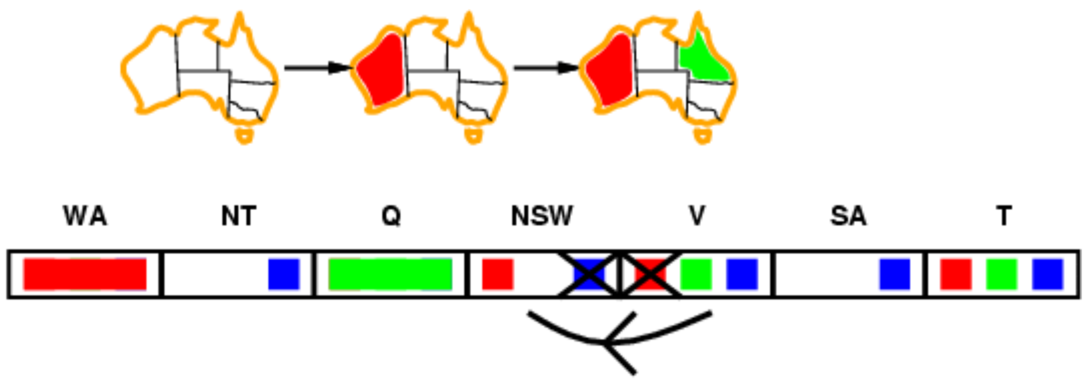


Arc consistency

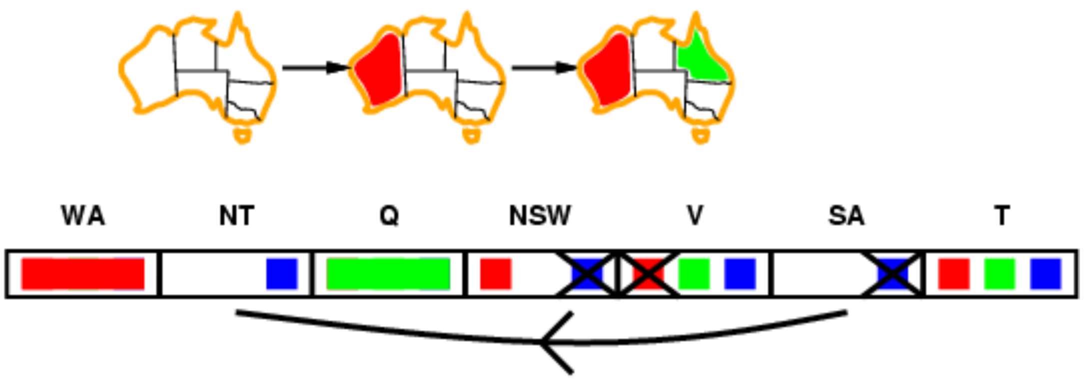
- Simplest form of propagation makes each arc consistent
  for every value *x* of *X* there is some allowed *y*



- If *X* loses a value, neighbors of *X* need to be rechecked

Arc consistency

Arc consistency detects failure earlier than forward checking and can be run as a preprocessor or after each assignment



Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if RM-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function RM-INCONSISTENT-VALUES( Xi, Xj) returns true iff remove a value
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy constraint(Xi, Xj)
         then delete x from DOMAIN[Xi];   removed ← true
   return removed
```

Time complexity: $O(n^2 d^3)$

Local search for CSPs

- *Hill-climbing* and *simulated annealing* typically work with "complete" states, i.e., all variables assigned

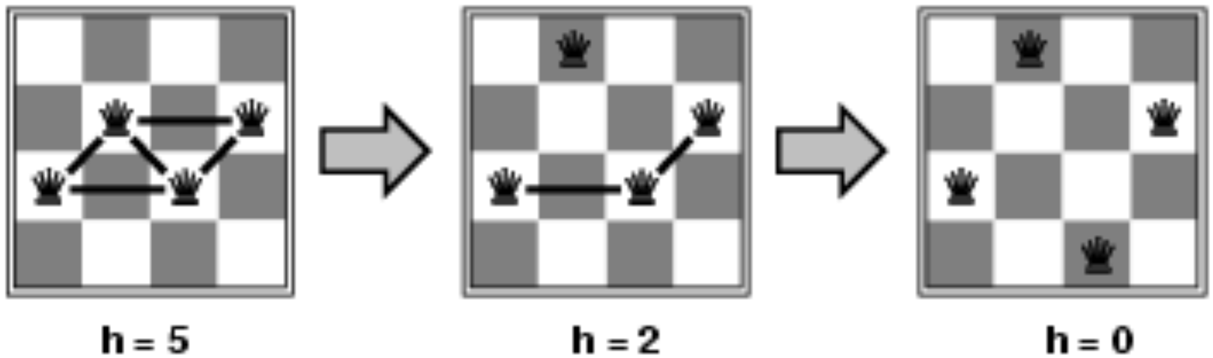
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks



- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice