Constraint satisfaction problems

- A Constraint Satisfaction problem consists of:
  - a set of variables
  - a set of possible values, a domain for each variable
  - a set of constraints amongst subsets of the variables
- The aim is to find a set of assignments that satisfies all constraints, or to find all such assignments.

Posing a Constraint Satisfaction Problem

A CSP is characterized by

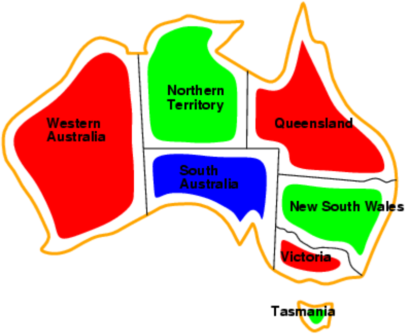
- A set of variables $V_1, V_2, \ldots, V_n$.
- Each variable V_i has an associated domain $dom(V_i)$ which specifies the set of possible values the variable can take. (We assume domains are finite.)
- A total assignment is an assignment of a value to each variable.
- A hard constraint on a subset of variables specifies which combinations of values are legal. The legal assignments are said to satisfy the constraint.
- A solution to CSP is total assignment that satisfies all the constraints.

Example: Map colouring

- Assign a colour (red, green, or blue) to each state so neighbouring states have different colours.
- What are the variables?
- What are the domains?
- How many total assignment are there?
- What are the constraints?

Example: Map colouring

Possible solution.



Simple Examples

Example 1:

- Variables: A, B, C
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B$, $B < C$

Example 2:

- Variables: A, B, C, D
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B$, $B < C, C < D$

Example 3:

- Variables: A, B, C, D, E
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B$, $B < C, C < D, D < E$

CSP variants

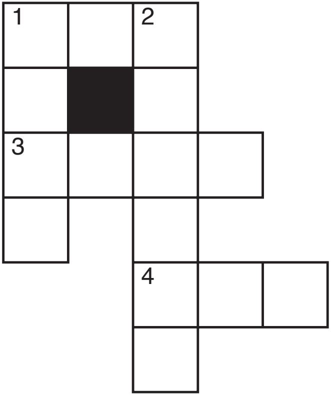
- determine whether or not a solution exists
- find a solution
- find all solutions
- count the number of solutions
- find the best solution given some solution quality
    - soft constraints specify preferences
- determine whether some property holds in all of the solutions

Example: scheduling activities

- Variables: A, B, C, D, E that represent the starting times of various activities.
- Domains: $dom(A) = \{1, 2, 3, 4\}$, $dom(B) = \{1, 2, 3, 4\}$, $dom(C) = \{1, 2, 3, 4\}$, $dom(D) = \{1, 2, 3, 4\}$, $dom(E) = \{1, 2, 3, 4\}$
- What are some total assignments?
- How many total assignments are there?
- Constraints:

$$
\begin{aligned}
&(B \neq 3) \wedge (C \neq 2) \wedge (A \neq B) \wedge (B \neq C) \wedge \\
&\quad (C < D) \wedge (A = D) \wedge (E < A) \wedge (E < B) \wedge \\
&\quad (E < C) \wedge (E < D) \wedge (B \neq D).
\end{aligned}
$$

Example: Crossword Puzzle

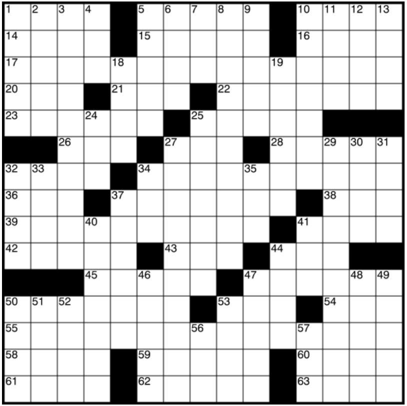


Words:

ant, big, bus, car, has
book, buys, hold,
lane, year
beast, ginger, search,
symbol, syntax

- What are the variables?
- What are their domains?
- How many total assignments are there?
- What are the constraints?

Example: Crossword Puzzle



Suppose there are 10,000 words of each length (from 2 to 10).

- How many total assignments are there?

Example: Sodoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- What are the variables?
- What is their domain?
- How many total assignments are there?
- What are the constraints?

Hard and Soft Constraints

- Given a set of variables, assign a value to each variable that either
  - satisfies some set of constraints: satisfiability problems — "hard constraints"
  - minimizes some cost function, where each assignment of values to variables has some cost: optimization problems — "soft constraints"
- Many problems are a mix of hard and soft constraints (called constrained optimization problems).

Learning Objectives

At the end of the class you should be able to:

- show how constraint satisfaction problems can be solved with generate-and-test
- show how constraint satisfaction problems can be solved with search
- explain and trace arc-consistency of a constraint graph
- show how domain splitting can solve constraint problems

Generate-and-Test Algorithm

- Generate the assignment space $\mathbf{D} = dom(V_1) \times dom(V_2) \times \ldots \times dom(V_n)$. Test each assignment with the constraints.
- Example:

$$\begin{aligned} \mathbf{D} &= dom(A) \times dom(B) \times dom(C) \times dom(D) \times dom(E) \\ &= \{1,2,3,4\} \times \{1,2,3,4\} \times \{1,2,3,4\} \times \{1,2,3,4\} \times \{1 \\ &= \{\langle 1,1,1,1,1 \rangle, \langle 1,1,1,1,2 \rangle, ..., \langle 4,4,4,4,4 \rangle\}. \end{aligned}$$

- Can be implemented with n nested for-loops.

```
for A in dom_A:
   for B in dom_B:
     ...
      if constraints are satisfied: return (A,B,...)
```

- How many assignments need to be tested for n variables each with domain size d?

Backtracking Algorithms

- Systematically explore **D** by instantiating the variables one at a time
- evaluate each constraint predicate as soon as all its variables are bound
- any partial assignment that doesn't satisfy the constraint can be pruned.

Example Variables A, B, C, domains $\{1, 2, 3, 4\}$,
constraints $A < B, B < C$.
Assignment $A = 1 \wedge B = 1$ is inconsistent with constraint $A < B$ regardless of the value of the other variables.

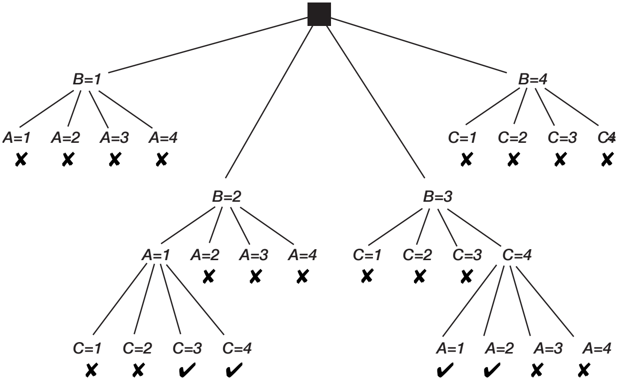
CSP as Graph Searching

A CSP can be solved by graph-searching:

- A node is an assignment values to some of the variables.
- Suppose node N is the assignment $X_1 = v_1, \ldots, X_k = v_k$.
  Select a variable Y that isn't assigned in N.
  For each value $y_i \in dom(Y)$
  $X_1 = v_1, \ldots, X_k = v_k, Y = y_i$ is a neighbour if it is consistent with the constraints that can be evaluated.
- The start node is the empty assignment.
- A goal node is a total assignment that satisfies the constraints.
- The search space depends on which variable is selected to be assigned for each node. There are no cycles or multiple paths to a node.

Simple Example 1

- Variables: A, B, C
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B,\ B < C$



Simple Example 2

- Variables: A, B, C, D
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B$, $B < C, C < D$

Simple Example 3

- Variables: A, B, C, D, E
- Domains: $\{1, 2, 3, 4\}$
- Constraints $A < B$, $B < C, C < D, D < E$

Example: scheduling activities

- Variables: A, B, C, D, E that represent the starting times of various activities.
- Domains: $dom(A) = \{1, 2, 3, 4\}$, $dom(B) = \{1, 2, 3, 4\}$, $dom(C) = \{1, 2, 3, 4\}$, $dom(D) = \{1, 2, 3, 4\}$, $dom(E) = \{1, 2, 3, 4\}$
- Constraints:

$$(B \neq 3) \land (C \neq 2) \land (A \neq B) \land (B \neq C) \land$$
$$(C < D) \land (A = D) \land (E < A) \land (E < B) \land$$
$$(E < C) \land (E < D) \land (B \neq D).$$

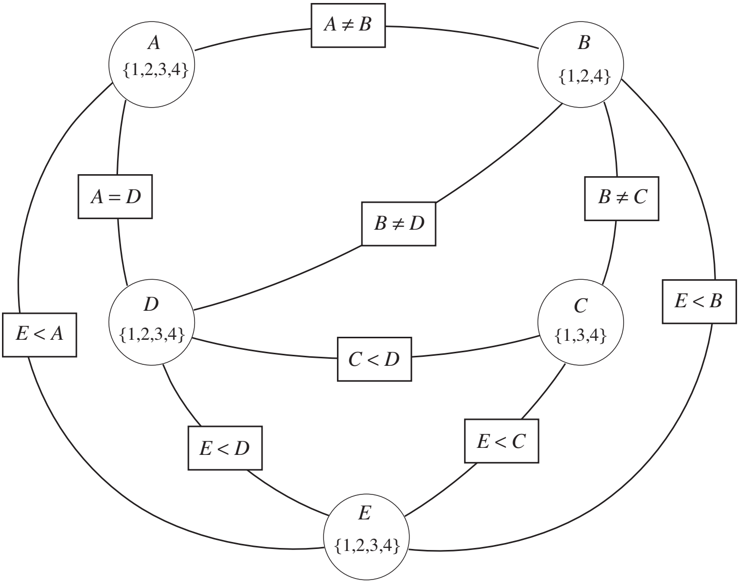
Consistency Algorithms

- Idea: prune the domains as much as possible before selecting values from them.
- A variable is domain consistent if no value of the domain of the variable is ruled impossible by any of the constraints.
- Example: Is the scheduling example domain consistent? $dom(B) = \{1, 2, 3, 4\}$ isn't domain consistent as $B = 3$ violates the constraint $B \neq 3$.

Constraint Network

- There is a oval-shaped node for each variable.
- There is a rectangular node for each constraint.
- There is a domain of values associated with each variable node.
- There is an arc from variable X to each constraint that involves X.
  An arc is written as $\langle X, r(X, \overline{Y}) \rangle$
  E.g., $\langle X, X < Y \rangle$, $\langle Y, X < Y \rangle$
  $\langle X, X + Y = Z \rangle$, $\langle Y, X + Y = Z \rangle$, $\langle Z, X + Y = Z \rangle$

Example Constraint Network



Arc Consistency

- An arc $\langle X, r(X, \overline{Y}) \rangle$ is arc consistent if, for each value $x \in dom(X)$, there is some value $\overline{y} \in dom(\overline{Y})$ such that $r(x, \overline{y})$ is satisfied.
- A network is arc consistent if all its arcs are arc consistent.
- What if arc $\langle X, r(X, \overline{Y}) \rangle$ is *not* arc consistent?
All values of X in $dom(X)$ for which there is no corresponding value in $dom(\overline{Y})$ can be deleted from $dom(X)$ to make the arc $\langle X, r(X, \overline{Y}) \rangle$ consistent.

Arc Consistency Algorithm

- The arcs can be considered in turn making each arc consistent.
- When an arc has been made arc consistent, does it ever need to be checked again?
  An arc $\langle X, r(X, \overline{Y}) \rangle$ needs to be revisited if the domain of one of the Y's is reduced.

Generalized Arc Consistency

for each variable X:
$D_X := dom(X)$
$to\_do := \{\langle X, c\rangle \mid c \in C$ and $X \in scope(c)\}$
while to_do is not empty:
select and **remove** path $\langle X, c\rangle$ from to_do
suppose scope of c is $\{X, Y_1, \ldots, Y_k\}$
$ND_X := \{x \mid x \in D_X$ and
exists $y_1 \in D_{Y_1}, \ldots, y_k \in D_{Y_k}$
s.th. $c(X = x, Y_1 = y_1, \ldots, Y_k = y_k) = true\ \}$
if $ND_X \neq D_X$:
$to\_do := to\_do \cup \{\langle Z, c'\rangle \mid X \in scope(c'),$
c' is not c, $Z \in scope(c') \setminus \{X\}\}$
$D_X := ND_X$
return $\{D_X \mid X$ is a variable$\}$

Arc Consistency Algorithm

Three possible outcomes when all arcs are made arc consistent:

- One domain is empty $\Longrightarrow$ no solution
- Each domain has a single value $\Longrightarrow$ unique solution
- Some domains have more than one value $\Longrightarrow$ there may or may not be a solution

Complexity of Arc Consistency

- Consider binary constraints
- Each variable domain is of size d
- There are e arcs.
- Checking an arc takes time $O(d^2)$
  $\langle X, c(X, Y) \rangle$ for each value for X, check each value for Y
- Each constraint needs to be checked at most d times.
  $\langle X, c(X, Y) \rangle$ rechecked when a value for Y is removed.
- Thus the algorithm GAC takes time $O(ed^3)$.

Solving a CSP is an NP-complete problem where n the number of variables

- Give a solution it can be checked in polynomial time
- But it can be made arc consistent in polynomial time. How?
  Making the network arc consistent does not solve the problem. We need to search for a solution.

Finding solutions with AC and domain splitting

To solve a CSP:

- Simplify with arc-consistency
- If a domain is empty, return no solution
- If all domains have size 1, return solution found
- Else split a domain, and recursively solve each half.

Finding one solutions with AC and domain splitting

$Solve\_one(CSP, domains)$:
    simplify CSP with arc-consistency
    if one domain is empty:
        return False
    else if all domains have one element:
        return solution of that element for each variable
    else:
        select variable X with domain D and $|D| > 1$
        partition D into D_1 and D_2
        return $Solve\_one(CSP, domains$ with $dom(X) = D_1)$ or
                $Solve\_one(CSP, domains$ with $dom(X) = D_2)$

Finding set of all solutions with AC and domain splitting

$Solve\_all(CSP, domains)$:
  simplify CSP with arc-consistency
  if one domain is empty:
    return $\{\}$
  else if all domains have one element:
    return {solution of that element for each variable}
  else:
    select variable X with domain D and $|D| > 1$
    partition D into D_1 and D_2
    return $Solve\_all(CSP, domains$ with $dom(X) = D_1) \cup$
      $Solve\_all(CSP, domains$ with $dom(X) = D_2)$

AC and domain splitting as search

Domain splitting leads to search space

- Nodes: CSP with arc-consistent domains
- Neighbors of CSP:
  if all domains are non-empty:
    select variable X with domain D and $|D| > 1$
    partition D into D_1 and D_2
    neighbors are
    - $make\_AC(CSP \mid dom(X) = D_1)$
    - $make\_AC(CSP \mid dom(X) = D_2)$
- Goal: all domains have size 1
- Start node: $make\_AC(CSP)$